1. Is it possible that an event is independent of itself? If so, when?
2. Is it always true that if A and B are independent events, then Ac and Bc are independent events? Show that it is, or give a counterexample.

Solution

Let's break down the questions one by one.

### 1. Is it possible that an event is independent of itself? If so, when?

An event \( A \) is independent of itself if the probability of \( A \) occurring is the same regardless of whether \( A \) has already occurred. Mathematically, this is expressed as:

\[

P(A \cap A) = P(A) \times P(A)

\]

But \( P(A \cap A) = P(A) \) (because the intersection of \( A \) with itself is just \( A \)).

So, the equation becomes:

\[

P(A) = P(A) \times P(A)

\]

This simplifies to:

\[

P(A) = P(A)^2

\]

This equation holds if either:

1. \( P(A) = 1 \)

2. \( P(A) = 0 \)

Thus, an event \( A \) is independent of itself only if \( P(A) = 0 \) or \( P(A) = 1 \). This means that the event either always happens or never happens.

### 2. Is it always true that if \( A \) and \( B \) are independent events, then \( A^c \) and \( B^c \) are independent events?

Yes, it is always true. Here's why:

Given that \( A \) and \( B \) are independent events, we know that:

\[

P(A \cap B) = P(A) \times P(B)

\]

We need to prove that \( A^c \) and \( B^c \) (the complements of \( A \) and \( B \)) are also independent.

We know that:

\[

P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B)

\]

Using the inclusion-exclusion principle:

\[

P(A \cup B) = P(A) + P(B) - P(A \cap B)

\]

Substituting the value of \( P(A \cap B) \):

\[

P(A \cup B) = P(A) + P(B) - P(A) \times P(B)

\]

Therefore:

\[

P(A^c \cap B^c) = 1 - [P(A) + P(B) - P(A) \times P(B)]

\]

Expanding the complement:

\[

P(A^c \cap B^c) = (1 - P(A))(1 - P(B))

\]

Notice that:

\[

P(A^c) = 1 - P(A)

\]

\[

P(B^c) = 1 - P(B)

\]

Thus:

\[

P(A^c \cap B^c) = P(A^c) \times P(B^c)

\]

This shows that \( A^c \) and \( B^c \) are independent events if \( A \) and \( B \) are independent.